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An Active-Learning Approach to Visualising Multivariate Functions using Balloons

Stephen E. Erfle
International Business and Management
Dickinson College, erfle@dickinson.edu

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An Active-Learning Approach to Visualising Multivariate Functions using Balloons

Stephen E. Erfle
International Business and Management
Dickinson College
erfle@dickinson.edu

Abstract

This hands-on multi-class exercise helps students with limited mathematical training better visualise how economists represent three-dimensional surfaces in two dimensions as indifference curves, isoquants and isoprofit contours using a balloon. Many students study microeconomics with limited formal mathematical training in analysing multivariate functions. Despite this weakness, the marginal analysis that is at the core of microeconomic analysis requires that we teach students to think in terms of marginal tradeoffs and to understand how these tradeoffs must be balanced to achieve an optimal outcome. One could use this exercise in introductory level classes by simply ignoring the calculus footnotes. The exercise protocol provides detailed instructions in PowerPoint slides for building the balloon as well as how to use the surface of the balloon to model economic concepts such as marginal product, the economic region of production, and marginal rate of technical substitution (and their consumer side counterparts). Instructions show students how to find and interpret a point where x is more valuable than y and to compare that with the reverse situation. This protocol concludes with instructions allowing students to analyse constrained optimisation visually including what the tangency condition implies regarding the marginal tradeoffs inherent in the constrained optimal solution.

Keywords: Graphing multivariate functions, level sets, gradient, constrained optimisation, microeconomic analysis

1. Introduction

This paper describes a multi-pronged approach to teaching students how to visualise multivariate functions. This paper provides support for the PowerPoint file that is the most efficient delivery mechanism to describe this exercise protocol to students. (For convenience, I call this the PPT file, and specific slides from that file are called PPT.#.) The PPT file provides detailed instructions for building the balloon model as well as how to use the surface of the balloon to model economic concepts such as marginal product, the economic region of production, and marginal rate of technical substitution (and their consumer side counterparts). Instructions show students how to find and interpret a point where x is more valuable than y and to compare that with the reverse situation. This protocol concludes with instructions allowing students to analyse constrained optimisation problems visually including a discussion of what the tangency condition implies regarding the marginal tradeoffs inherent in the constrained optimal solution.

This paper discusses the overall structure of the exercise protocol and highlights some of the most important parts of the protocol in detail, rather than describing the entire protocol, slide by slide. This discussion is targeted to a student audience so that they have another point of reference for understanding the material. Slides in the PPT file have notes for instructors, however, the slides were created for students to view through overhead projection. Extensions and algebraic counterparts are in appendices to the PPT file and in separate annotated instructor Excel and Word documents.

2. Exercise structure

This is a three-part, hands-on, in-class exercise that is designed to help you think about how economists represent 3-dimensional surfaces in two dimensions (as isoquants or isoprofit contours, for example) using a balloon. By creating local representations of a topographic map of your own balloon, you should be able to better visualise various aspects of analysing 3-dimensional surfaces. You can use the surface of the balloon to model economic concepts such as marginal product, the economic region of production, and marginal rate of technical substitution, and their consumer-side counterparts. You will learn how to find and interpret a point where x is more valuable than y and to compare that with the reverse situation. This protocol concludes with instructions allowing you to analyse constrained optimisation visually including what the tangency condition implies regarding the marginal tradeoffs inherent in the constrained optimal solution.

3. Building your balloon model (PPT.3 – PPT.5)

Before attacking the three analytical tasks outlined on PPT.2, you must first undertake the mechanical task of building your model. PPT.3 describes what you will need to build and analyse your balloon model. There are two versions of the model, A-style and B-style, and every other person should have a different style balloon (and slightly different instructions in some places as a result). Both styles have different economic interpretations and by looking at yours, and at your neighbour's, you will learn more than by analysing just your own balloon. Make sure both pieces of graph paper look the same. (Note to instructors: download the graph paper using the **CLICK for HANDOUT** hyperlink on PPT.3 which provides a set of graph paper pages for each balloon style.) Each piece of graph paper has three "orientation guides." Each orientation guide shows the analytical tasks for that part of the exercise using lines that have been pre-drawn on the graph paper.

Detailed instructions on how to build the model are found on PPT.4 and PPT.5. You have two tasks: blowing up the balloon, found on PPT.4, and attaching it to the base, found on PPT.5. This project requires oblong balloons, however round balloons are the only kind that are readily available. Luckily, you can turn a round balloon into an oval one if you blow it up while restricting its middle (I realise that sounds strange so the best way to see how to do this is to watch the 21-second YouTube video by clicking

Watch Here on PPT.4). You want an oval that is about the size of the graph paper (or a bit smaller) when you place it on the graph paper and view it from above. The most interesting economic interpretation occurs if your oval has a tilt when viewed from above. (It is often easier to talk about the graph paper using North, South, East, and West designations rather than using coordinates so the origin is the SW corner of the page.) Two orientations are shown in the mock-ups in the lower right of Figure 1:

A-style) Positive long-axis tilt starting near the origin (SW) and angling to the NE

B-style) Negative long-axis tilt starting in the NW and angling to the SE

Both styles of balloons have interesting economic interpretations. It turns out that A-style balloons model imperfect complements and B-style balloons model imperfect substitutes (in both a consumption and production sense) in the economic region (the "3rd quadrant relative to the top of the balloon" that will be discussed shortly).

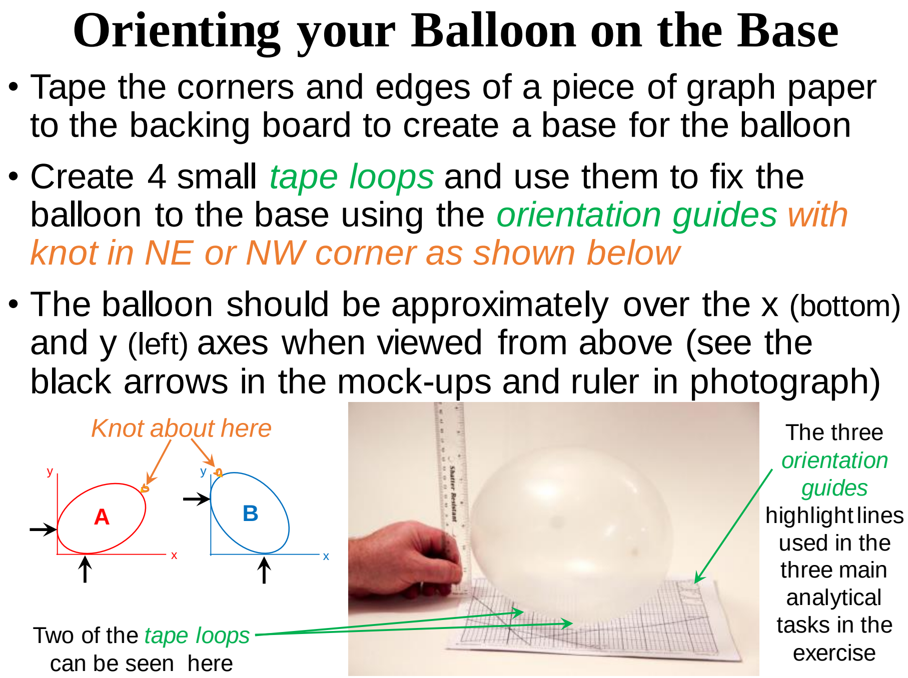

Figure 1: Connecting the balloon to the base (PPT.5)

Figure 1 describes how and where to attach the balloon to the base. Create the base by attaching one of the pieces of graph paper to the base using 8 pieces of tape, one at each corner and one at the middle of each side. Put the base in landscape and put your name on the base and on the graph page in the upper right corner next to the word *name*. If your balloon has a "flatter" side, anchor it to the graph paper base as this will provide the most stable model to work from in the end. Place the balloon diagonally on the paper with the tied-off-end in the NE for A-style balloons and in the NW for B-style balloons. (We will be working on the SW and SE parts of the balloon in our analysis and we want smooth surfaces there.)

4. Analytical task 1: Visualising marginal value of x and marginal value of y, and using them to find the top of a hill

For all exercises, consider only the top half of the balloon – that part of the balloon that curves toward the "top" (not the underside that curves down to where the balloon is anchored to the graph-paper base by tape loops). We can use the top half to model a number of economic situations:

1. A bliss point in consumer theory over two goods x and y;
2. A production function in producer theory over two inputs such as labour (x) and capital (y);
3. A profit hill for a firm producing two goods x and y.

In the first instance, the height above the (x, y) graph paper, z, is utility; in the second, z is output of a good produced; and in the third, z is profits. In each instance, we can think of this height as coming from a multivariate function $z = z(x, y)$ although you may not know the actual algebraic form of that function.

In contrast to univariate functions like $y = y(x)$ which have only one slope, $\Delta y/\Delta x$, multivariate functions like $z = z(x, y)$ have slopes in different directions. If you consider the hiking analogy and you are on a nice smooth hill, you will likely face different slopes if you walk in different directions. Of course, when hiking you can talk about the slope as you move in any direction of the compass. In the present context, two directions are of interest. The first involves varying x while holding y fixed (moving from W to E holding the N/S coordinate fixed) and the second involves varying y while holding x fixed (moving from S to N holding the E/W coordinate fixed).[1] The interpretation of a slope in one direction holding the other direction fixed follows the economist's strategy of "holding all else constant" (which economists often state in Latin as *ceteris paribus*) when they perform a comparative statics analysis.

4.1. Visualising slope in the x and y direction (PPT.6 and PPT.7)

First, hold y fixed and consider the z slope in the x direction, $\Delta z/\Delta x$. ***This slope is called the marginal value of x, MV_x.*** Start with your ruler in a vertical position with $y = 10$ on the vertical axis and move the ruler to the right (from W to E along $y = 10$) until you reach the vertical edge of the balloon as seen in the left photo in PPT.6, shown as Figure 2.[2] Looking at the model from the side (from S to N), roll the ruler edge from W to E along the surface of the balloon moving from vertical (slope of $+\infty$) slope over the entire

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[1] In terms of calculus, the first is the partial derivative of z with respect to x, $\partial z/\partial x$, and the second is partial derivative of z with respect to y, $\partial z/\partial y$.

[2] The tangency between balloon and ruler is a point where $\partial z/\partial x = +\infty$, given $y = 10$.

range of positive slopes to horizontal (slope of 0) then over the range of negative slopes back to vertical (slope of -∞) on the Eastern edge of the balloon while keeping y = 10.[3] Notice that the slope in the x direction smoothly decreased from +∞ to -∞ as x increased from W to E. Note that if you use a different value of y, the slope for any given x would change.[4]

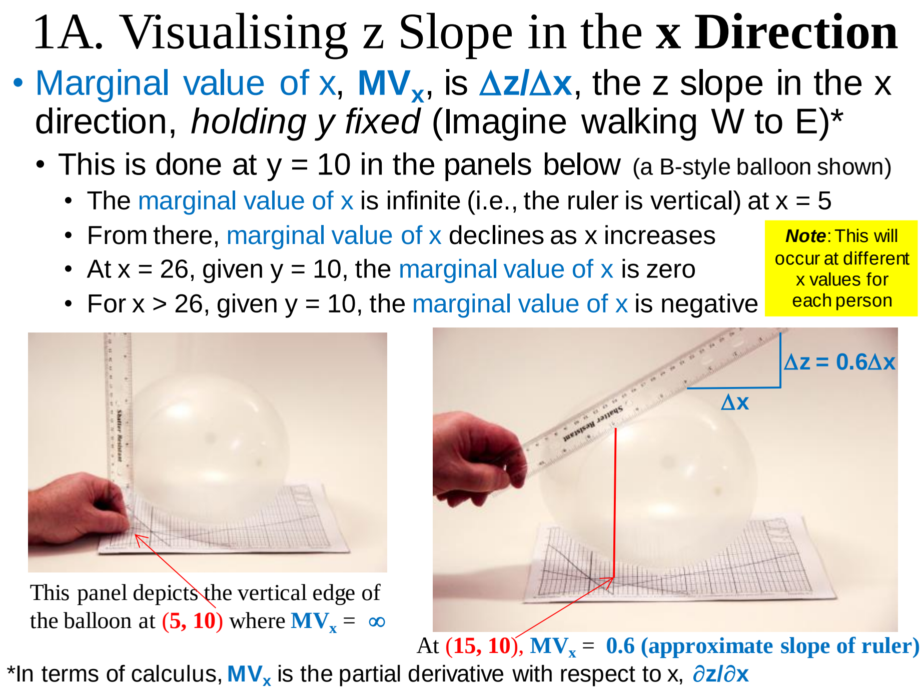

Figure 2: Visualising z slope in the x direction (PPT.6)

Next, hold x fixed and consider the z slope in the y direction, Δz/Δy. ***This slope is called the marginal value of y, MV_Y.*** Start with your ruler in a vertical position with x = 10 on the horizontal axis and move the ruler up (from S to N along x = 10) until you reach the vertical edge of the balloon as seen in the left photo in PPT.7.[5] Looking at the model from the side (from E to W so that y values increase moving left to right), roll the ruler edge from S to N along the surface of the balloon moving from vertical slope (slope of +∞) over the entire range of positive slopes to horizontal (slope of 0 shown in the right photo in PPT.7) then over the range of negative slopes back to vertical (slope of -∞) on the Northern edge of the balloon (right given you are looking at this from E to W)

[3] Each of these slopes is a partial derivative of z with respect to x, ∂z/∂x, given y = 10.

[4] Mathematically, the ∂z/∂x depends on the value of y that is being held fixed.

[5] The tangency between balloon and ruler is a point where ∂z/∂y = +∞, given x = 10.

while keeping x = 10.[6] Notice that the slope in the y direction smoothly decreased from +∞ to -∞ as y increased from S to N. Note that if you use a different value of x, the slope for any given y will change.[7]

4.2. Finding the top of a hill using MV = 0 lines (PPT.8 and PPT.9)

If you are out hiking, how do you know you are at the top of a nice smooth hill? The answer is that your feet are flat in whatever direction you turn. We can use MV = 0 lines to indirectly find the top of a hill. Figure 3 shows how the endpoints of both of the MV = 0 line (segments) are found in PPT.8.

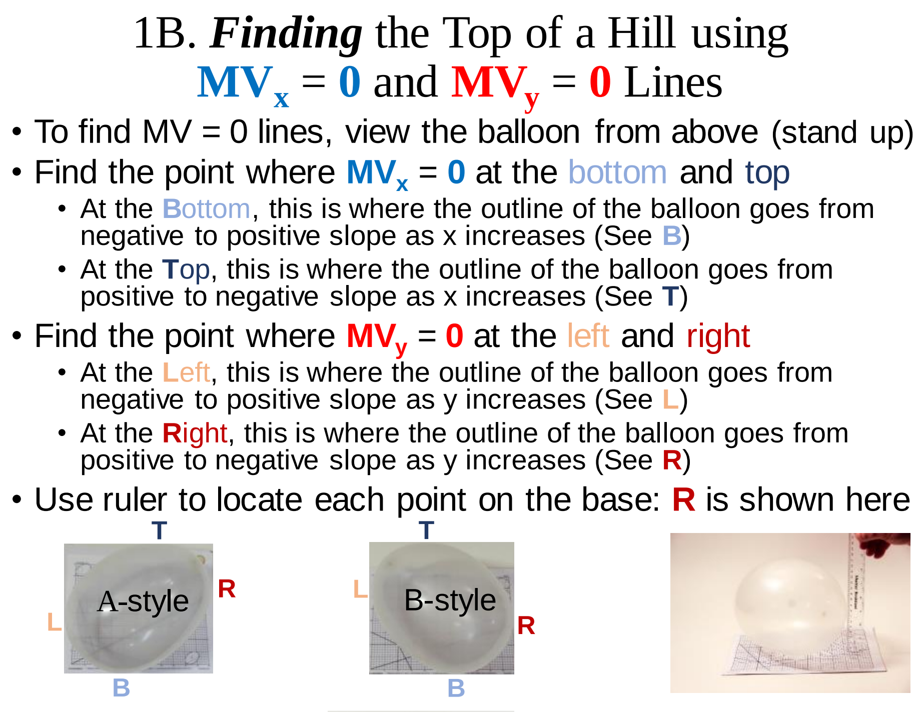

Figure 3: Finding points where $MV_X = 0$ and $MV_Y = 0$ (PPT.8)

The important point in finding MV = 0 points is to view the balloon from above. The actual location of the four endpoints (**B**, **T**, **L** and **R**) will differ from person to person because of a) balloon style differences, b) balloon size differences, and c) balloon placement on the base differences. Do not worry if one or more of these points are off the graph paper when viewed from above. So long as you can approximately locate the endpoint (for example, endpoint **T** is 1 inch above the top of the paper with an x

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[6] Each of these slopes is a partial derivative of z with respect to y, $\partial z/\partial y$, given x = 10.

[7] Mathematically, the $\partial z/\partial y$ depends on the value of x that is being held fixed.

value of x = 33), then this information can be transferred to the graph page as discussed in PPT.9 shown here as Figure 4.

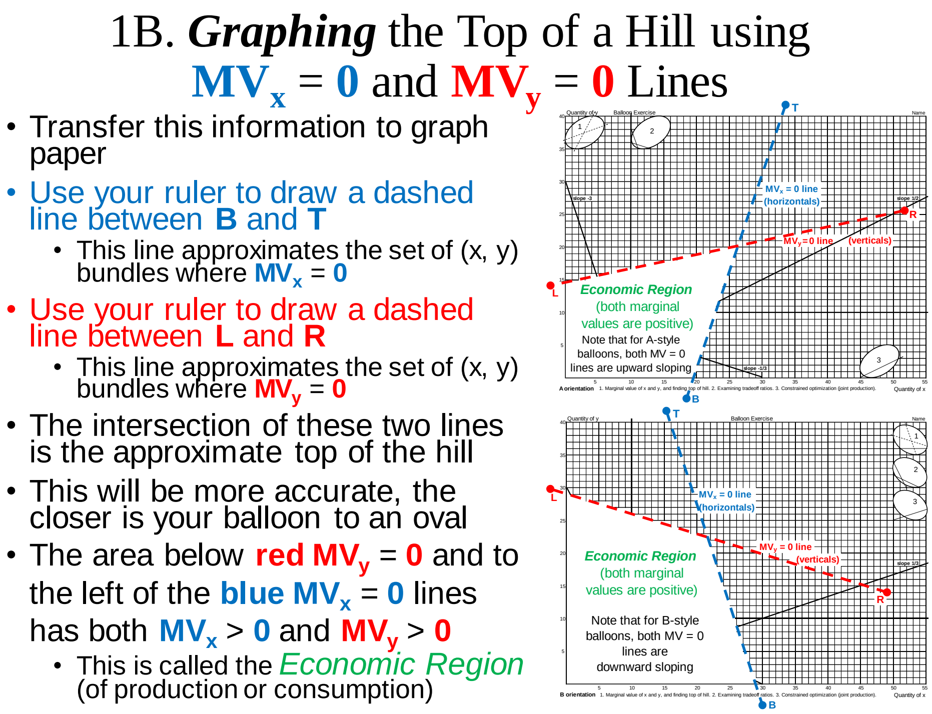

Figure 4: Graphing points where MVX = 0 and MVY = 0 to find the top of a hill (PPT.9)

To the extent that the balloon is a symmetric oval when viewed from above, the intersection of the blue MVX = 0 and red MVY = 0 lines drawn in Figure 4 should closely approximate the top of the hill (if the oval is not symmetric, the marginal value equals zero lines are curves). Verify this by viewing the balloon from the bottom (S to N) to determine the x coordinate of the top of the hill and by viewing from the side (W to E) to determine the y coordinate of the top of the hill. Notice that, if you are at the top of a smooth hill, your feet are flat (MV = 0) whether your feet are pointed in the East (x) or the North (y) direction. This is exactly what you would expect to happen at the intersection of the two MV = 0 lines.

The MV = 0 lines create four "quadrants" relative to the top of the balloon. The area to the SW of the top (i.e., the area closest to the origin) is upward sloping in both the x and y direction. This "3rd quadrant relative to the top of the balloon" has a less than 90° angle for A-style balloons and has a more than 90° angle for B-style balloons.

5. Analytical task 2: Level sets, tradeoff ratios and their relation to MVx and MVY

If you are hiking and you are on the side of a hill, there will be a slope in all directions of the compass. One task on the first day was to consider slope in two specific directions, x and y. Now we can use those slopes to describe the change in z that occurs in ANY direction. If we consider a move of Δx in the x direction and Δy in the y direction, then the change in z in the (Δx, Δy) direction is approximately given by:[8]

$$\Delta z = \Delta x \cdot MV_X + \Delta y \cdot MV_Y \qquad (1)$$

5.1. Level sets and tradeoff ratios

The change in z in Equation 1 may be positive, negative or zero. A large part of what we will be doing in this exercise is to analyse properties of *level sets* of the balloon. A level set is simply a set of points where $\Delta z = 0$ in Equation 1 so that the z coordinate is the same height, just like a contour line on a topographic map. A level set will have a different interpretation in each of the economic models 1 – 3 discussed at the start of Section 4. In the consumer theory context, level sets are called *indifference curves*. In the producer theory context, they are called *isoquants*, and in the profit hill context, they are called *isoprofit contours*. At a semantic level, they are all isoquants because isoquant means equal quantity: in the first instance, that quantity is utility; in the second, output; and the third, profits.

The economic interpretation of minus the (x, y) slope of a level set at a point, $-\Delta y/\Delta x$, in the 3rd quadrant varies in each of the three economic models. The first is conceptualised as the *marginal rate of substitution*, MRS, and it is the rate at which y can be exchanged for x to maintain the same utility level. The second, often called the *marginal rate of technical substitution*, MRTS, (some authors call this MRS, RTS, or TRS), is the rate at which one input in the production process, y – think capital, can be substituted for another input, x – think labour, to maintain the same level of output produced. The third describes the rate at which y can be exchanged for x by a firm producing both x and y while maintaining profits at a given level. Each is a ***tradeoff ratio***. In each instance, the tradeoff ratio is the ratio of marginal values (set $\Delta z = 0$ and solve for $-\Delta y/\Delta x$ in Equation 1):[9]

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[8] This, is simply the discrete version of the differential of z: $dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$. The discrete change in Equation 1 is only an approximation because slopes change on a curve. This approximation is better for small changes in x and y.

[9] Formally, the ratio is: Tradeoff ratio = $\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$. This follows from the fact that along a level set, the differential of z, dz, is zero: $dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy = 0$. Solving for minus the slope, $-\frac{dy}{dx}$, we obtain the tradeoff ratio.

$$\text{Tradeoff ratio} = \frac{\text{Marginal value of x}}{\text{Marginal value of y}} = \frac{MV_X}{MV_Y} \quad (2)$$

The slope of the level set is negative in the 3rd quadrant relative to the top of the balloon meaning that less y is used when more x is used if z is held constant because, as noted above, each has positive marginal value. Economists typically strip the minus sign and talk in terms of positive numbers when discussing these tradeoffs. Economists call this 3rd quadrant the *economic region of consumption* in the first instance and the *economic region of production* in the second.

When the tradeoff ratio is larger than one in the 3rd quadrant, x has greater marginal value than y in "producing" z. The reverse holds true when the slope is less than one in the 3rd quadrant. Equation 2 makes this algebraically clear, but to visualise these comparisons graphically, it is useful to work with reciprocals (like 2 and ½ or 3 and 1/3) in this instance. For practical purposes, different tradeoff ratios for A-style and B-style balloons are employed in this exercise (because 2 and ½ are quite close to one another in A-style balloons and 3 and 1/3 are quite far apart in B-style balloons).

5.2. *Finding*, *Marking*, and *Graphing* a point where x is more valuable than y (and *vice versa* using PPT.12 – PPT.17)

When the tradeoff ratio is larger than one, x is more valuable than y (at the margin). This situation is analysed in a three-step procedure in PPT.12 – PPT.14. Figure 5 depicts the first step, *finding* a point where x is more valuable than y. This is a point with a high tradeoff ratio, **H**.

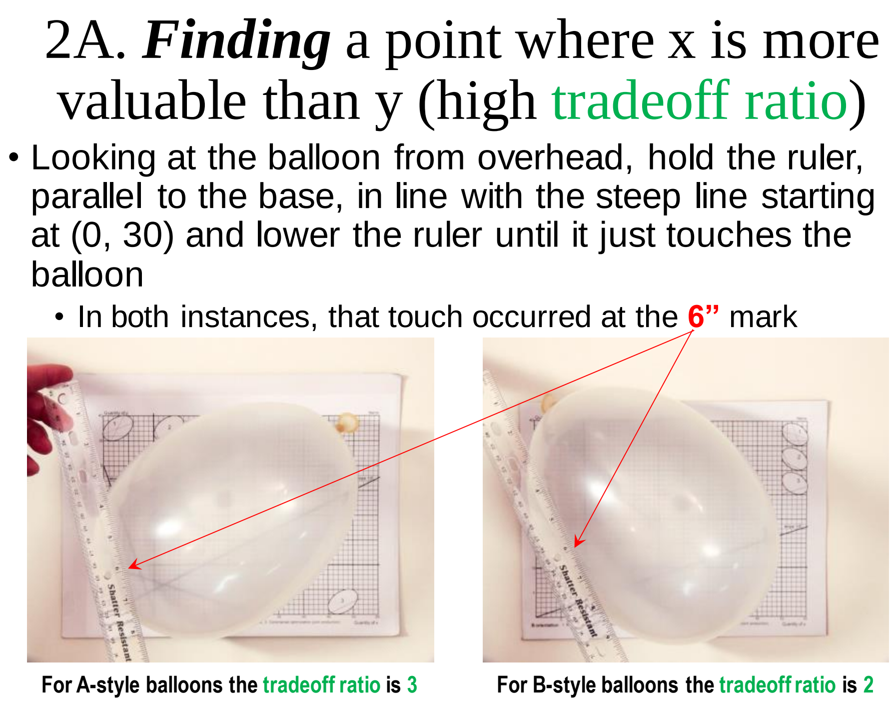

Figure 5: *Finding* a point where x is more valuable than y (PPT.12)

The next two steps, *marking* and *graphing* (in PPT.13 and PPT.14), describe how to mark this point on the balloon and then transfer that information onto the graph paper. Symmetric instructions in PPT.15 – PPT.17 walk you through finding, marking, and graphing a point with a low tradeoff ratio, **L**.

The balloon/ruler touch need not occur at the 6″ mark as shown in Figure 5. I adjusted my ruler so that I could easily identify the location of the tangency in the pictures. This is much easier to do if you use clear balloons because you can see the line that has been provided on the graph paper through the balloon. The steeply sloped line used to find **H** starts on the y axis at (0, 30) and has a slope of -3 for A-style and -2 for B-style balloons. The shallowly sloped line used to find **L** starts on the x axis at (30, 0) and has a slope of -1/3 for A-style and -1/2 for B-style balloons. Once you have found, marked, and graphed both **H** and **L** you are ready to consider how the MV_X and MV_Y at these points relate to one another.

5.1. Decomposing the direction of maximal change using MV_X and MV_Y

When you marked **H** and **L** on the balloon, you were asked to use a ⊥ mark. The bottom of the "inverted T" is a local representation of the level set at the point. If you think of the perpendicular part (the shaft of the "inverted T" as an arrow starting at the point **H** (or **L**), then the coordinates of the arrow are (MV_X, MV_Y) and that arrow is called the *gradient* at the point. We explore this concept in Figure 6.

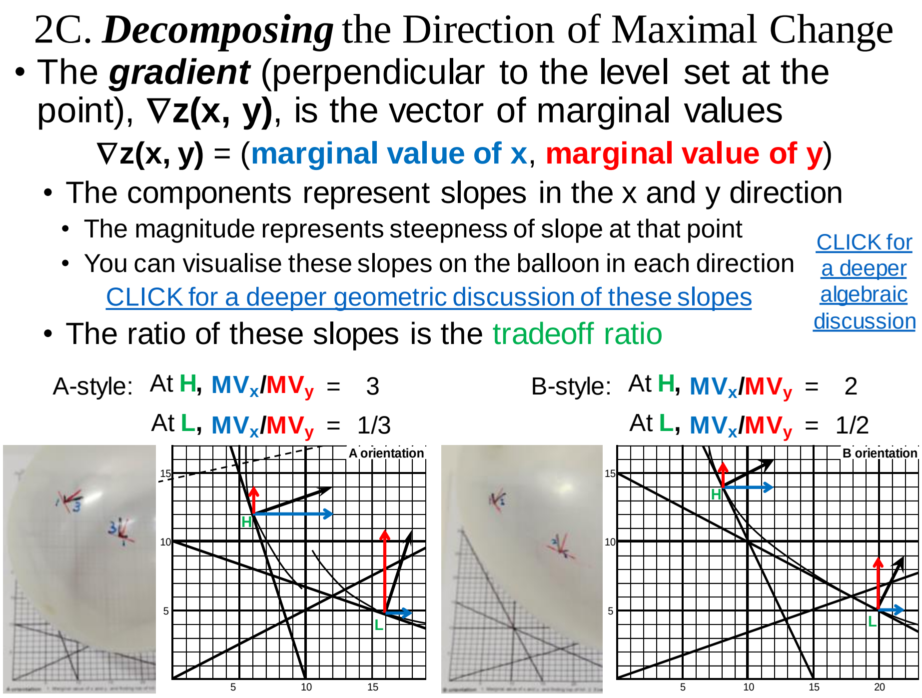

Figure 6: *Decomposing* the direction of maximal change (PPT.18)

Notice that the ratio of marginal values is the tradeoff ratio. You, of course, know that from Equation 2 which defined the tradeoff ratio, but here you can see the geometric meaning of this statement. Consider an A-style balloon at point **H**. At **H**, the MV_X is three times as large as the MV_Y meaning three times as steep. Different A-style balloons will have different steepnesses in each direction at **H**, but all will face this common *ratio* of slopes. Alternatively, consider a B-style balloon evaluated at point **L**. The slope in the x direction is half as large as in the y direction at point **L**. If you want to see additional detail on this geometric assertion, click on the hyperlink labelled **CLICK for a deeper geometric discussion of these slopes** which links you to slides PPT.34 and PPT.35.

In the *marking* slides PPT.13 and PPT.16, I asserted that the gradient, which is perpendicular to the base of the inverted T, represented the direction of maximal change. If you want to see a calculus-based proof of this assertion, click on the hyperlink labelled **CLICK for a deeper algebraic discussion** which links you to PPT.36.

6. Analytical task 3: Visualising constrained optimisation (PPT.19 – PPT.23)

Sometimes you cannot obtain an unconstrained optimum. Put another way, you face constraints on your behaviour. A wide variety of constraints are possible, but the easiest to depict are linear constraints such as a budget or time constraint, or a joint production constraint. PPT.19 discusses constraints in general and introduces the joint production constraint analysed in the next three slides. Joint production is examined in the same three-step fashion shown in Section 5.2 using the equation y = x/c (c units of x is produced for every unit of y). Figure 7 shows the second and third steps, marking the tangency point **C** on the balloon and graphing it on the graph paper based on the constraint y = x/2 for A-style and y = x/3 for B-style balloons. The tangency at **C** is decomposed into component marginal values as was done at **H** and **L** in Figure 6.

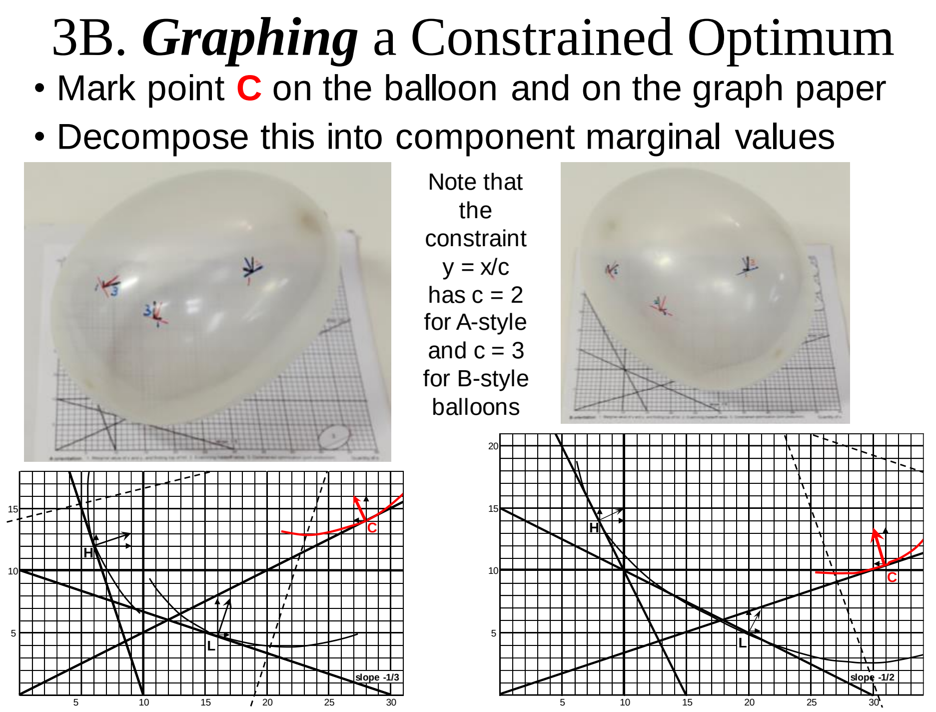

Figure 7: *Graphing* a constrained optimum involving joint production (PPT.21)

Notice that the tangency between balloon and ruler is now in the 4th quadrant relative to the top of the balloon. In this quadrant, the marginal value of x is negative, and the marginal value of y is positive. This is verified by analysing the component parts of the gradient as described in Section 5.3. Figure 8 shows that analysis.

3B. ***Analysing*** this Constrained Optimum

- The constrained optimum is no longer in the Economic Region
  - The constrained optimum is in the 4th quadrant relative to the top of the hill
  - In this region, $MV_x < 0$ and $MV_y > 0$ and the gradient points to the NW rather than NE as it did in Part 2
  - In this context, increasing x decreases profit but increasing y increases profit
- At the constrained optimum, the gain in profit from increased y should balance off the loss in profit from increased x
- Since each unit of y produces c units of x, $\Delta x = c \cdot \Delta y$
- In terms of the equation we initial examined,

$$0 = \Delta x \cdot MV_x + \Delta y \cdot MV_y = c \cdot \Delta y \cdot MV_x + \Delta y \cdot MV_y$$

0 = *Loss due to x + Gain due to y*

- Reorganizing and dividing by Δy we obtain: $MV_y = -c \cdot MV_x$
- In words, one unit of y must be c times as valuable as the loss from one unit of x because each unit of y comes with c units of x
- Geometrically, the slope in the y direction is c times as steep as in the x direction

Figure 8: *Analysing* a constrained optimum involving joint production (PPT.22)

The gain from an additional unit of y, MV_Y, must be c times as much as the loss from an additional unit of x, MV_X, because each unit of y produces c units of x. Geometrically, the slope at **C** in the y direction is c times as steep as the slope in the x direction.

Consider what happens just below and just above **C** = (x_c, y_c) on this line. If x and y are just below (x_c, y_c) then the losses from a bit more x are less than the gains from a bit more y. Profits will increase if more x and y are produced and sold in this instance. In terms of slopes, the z slope in the x direction, MV_X, will be smaller than at **C** and the z slope in the y direction, MV_Y, will be larger than at **C**. The reverse holds true above **C**. If x and y are just above (x_c, y_c) then the losses from a bit more x are more than the gains from a bit more y. Profits will increase if less x and y are produced and sold. In terms of slopes, MV_X will be larger than at **C** and MV_Y will be smaller than at **C**.

The strategy for analysing constrained optimisation discussed above heavily utilises the methodology presented in Sections 5.2 and 5.3 for analysing points where x is more valuable than y and *vice versa*. It should come as no shock to find that points **H** and **L** can be conceptualised as solutions to constrained optimisation problems. Three versions of each problem are proposed in PPT.23, shown here as Figure 9.

3C. Additional Constrained Optimisation Problems

- You have already found solutions to six other constrained optimisation problems in 2A-2C as tradeoff ratio tangencies between ruler & balloon

H is the solution to these three constrained optimisation problems:

At **H**, the tangency is $MV_x/MV_y = P_x/P_y$

*In this instance, P_x depends on balloon style

Maximise*	Subject to:[1]	Style: A	B		A	B
z = Utility(x, y)	Income = \$90	P_x = \$9	\$6	P_y = \$3	3	2
z = Output(labour, capital)	Cost = \$60	P_{labour} = \$6	\$4	$P_{capital}$ = \$2	3	2
z = Profit(x, y)	$Time_{total\ hours}$ = 30h	Time/x = 3h	2h	Time/y = 1h	3	2

[1]Each constraint can be written as y = 30 - 3x for A-style & y = 30 - 2x for B-style prices and income.

L is the solution to these three constrained optimisation problems:

At **L**, the tangency is $MV_x/MV_y = P_x/P_y$

**In this instance, P_y depends on balloon style

Maximise**	Subject to:[2]	Style: A	B		A	B
z = Utility(x, y)	Income = \$90	P_y = \$9	\$6	P_x = \$3	1/3	1/2
z = Output(labour, capital)	Cost = \$60	$P_{capital}$ = \$6	\$4	P_{labour} = \$2	1/3	1/2
z = Profit(x, y)	$Time_{total\ hours}$ = 30h	Time/y = 3h	2h	Time/x = 1h	1/3	1/2

[2]Each constraint can be written as y = 10 - x/3 for A-style & y = 15 - x/2 for B-style prices and income.

In each instance, the tangency condition can be recast as *equal bang for the buck* : $MV_x/P_x = MV_y/P_y$.

- You can readily create additional problems and find their solutions on the balloon by changing prices or income in the above constrained optimisation problems

Figure 9: Points **H** and **L** as solutions to constrained optimisation problems (PPT.23)

In this conceptualisation, the tangency condition that produced points **H** and **L** are:

$$\frac{MV_X}{MV_Y} = \frac{P_X}{P_Y} \qquad \text{Tradeoff ratio = price ratio} \qquad (3)$$

By cross-multiplying, Equation 3 can be rewritten as *the equal bang for the buck rule*:

$$\frac{MV_X}{P_X} = \frac{MV_Y}{P_Y} \qquad \text{Marginal value per dollar spent is equal across goods} \qquad (4)$$

This rule is commonly taught as the solution to the constrained optimisation problem in introductory microeconomics classes. In the consumer theory context, this is the equal marginal utility per dollar spent rule for choosing among goods and in the producer theory context it is the equal marginal product per dollar spent rule for choosing among factors of production.

It is worth noting that the price ratio, P_x/P_y, not the price of any one good, is what determines relative valuation. That is why I used different prices for x and y in the examples in Figure 9. The constraint function is Equation 5:

$$T = P_x \cdot x + P_y \cdot y \qquad \text{where T is the total (income, cost, or time) available} \qquad (5)$$

The three constraints which have **H** as the constrained optimum all have the same graph. This solution is seen by putting Equation 5 in slope intercept form. Note that this is the steeply sloped line passing through (0, 30). Similarly, the three constraints which have **L** as the constrained optimum all have the same graph which is the shallowly sloped line passing through (30, 0).

We can analyse other constraints in the same way. To test your comprehension of what you have learned, three scenarios are suggested, one for each of the conceptual models described at the start of Section 4. Footnotes provide answers for each scenario.

I. Suppose you wish to maximise utility over goods x and y but face a budget constraint. The price of x is \$2/unit, the price of y is \$3/unit, you have \$60 to spend on x, and y. (If you can afford the top of the hill given these prices with \$60 then imagine you only have \$30.) How much of each should you choose in order to maximise utility? What is MRS and what is the relation between the marginal utility of x and the marginal utility of y?[10]

II. Suppose you wish to maximise production of z using inputs labour (L, on the x axis) and capital (K, on the y axis) but face a cost constraint. The price of x (labour) is \$3/unit, the price of (capital) y is \$2/unit, you have \$60 to spend on x and y. How much of each should you choose in order to maximise output level z? What is MRTS and what is the relation between the marginal product of labour and the marginal product of capital?[11]

III. Suppose, you wish to maximise profits but face a capacity constraint of only being able to produce a total of 30 units of x and y per period. You could produce all x and be at the point (30, 0), all y and be at the point (0, 30), or produce somewhere in between and be at the point $(x_o, 30-x_o)$ where $0 < x_o < 30$. What production point maximises profits? What can we say about the marginal profit of x and the marginal profit of y at this point? Why?[12]

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[10] Each person's balloon will have a different point where this occurs but for each person, the point chosen will spend all their income. Equation 5 simplifies to, $2 \cdot x + 3 \cdot y = 60$ which, in slope-intercept form is $y = 20 - 2/3 \cdot x$. (If the top of your hill was affordable with \$60 in income, the \$30 in income version would have a budget constraint in slope-intercept form of $y = 10 - 2/3 \cdot x$.) The point chosen must be tangent. It therefore must satisfy the equation $MRS = P_x/P_y = 2/3$. Since MRS is the ratio of marginal utilities, the marginal utility of x must be two thirds the size of the marginal utility of y, $\partial z/\partial x = 2/3 \cdot \partial z/\partial y$. The reason is straightforward: in order to have equal bang for the buck across goods, MU_i/P_i must be equal across goods for utility maximization. This is the same condition as $MRS = P_x/P_y$.

[11] Each person's balloon will have a different point where this occurs but for each person, the firm spend \$60 on L (x) and K (y). The firm must be on the isocost line $3 \cdot L + 2 \cdot K = 60$ based on Equation 5 which in slope-intercept form is $y = 30 - 1.5 \cdot x$. The point chosen must be a tangency between isoquant and isocost so MRTS = 3/2. Thus, the marginal product of labour is 1.5 times the size of the marginal product of capital.

[12] Each person's balloon will have a different point where this occurs but for each person, the point chosen will have $x + y = 30$ based on Equation 5 and have equal marginal profit for both factors of production. The reason is simple: the tradeoff ratio is 1:1 in this instance.

7. Extensions of this analysis

The PPT file concludes with 18 additional slides. PPT.24 provides hyperlinks to the twelve appendices, three of which are discussed in Section 5.3. The final six slides provide instructor notes for the slides in the presentation and in the appendices. (When a pdf version of the PPT file is saved, only the slides are saved. If you want to distribute as a pdf file, you can delete slides you do not want to post (such as the algebraic slides with answers) prior to saving as a pdf file for distribution or posting to students.)

The above analysis is strictly geometric in nature. The algebra involved in describing ellipsoids is more complex than that used to describe paraboloids, and little is gained by the added complexity involved in algebraically modelling ellipsoids. As a result, I use equations for paraboloids to link the geometric materials in this exercise to algebraic problems. Four appendix slides provide algebra-based questions that link to the geometric analysis provided above. These questions formally require a knowledge of multivariate calculus, but I have found that partial derivatives can be taught to those without that formal training. An instructor-annotated Excel file and Word document supplement these algebraic questions.

One appendix slide, PPT.27, examines why A-style balloons model complements, and B-style balloons model substitutes in the economic region of consumption. This analysis is based on the decomposition of the *total effect* of a price change into the *substitution effect* and *income effect*. This slide also includes a discussion of *compensating variation* and *equivalent variation* of a price change that can be examined in this context. Six of the appendix slides show Excel-generated level set maps that allow you to show what complete level sets would look like instead of the local sketches that are provided in the main body of the presentation.