"But My Physics Teacher Said . . ."

A Mathematical Approach to a Physical Problem

By Jeffrey Forrester, Jennifer Schaefer, and Barry Tesman

This past semester I (Jennifer Schaefer) taught single-variable calculus. Like many of you, I have taught this course numerous times, so I have a good collection of examples, lecture notes, homework questions, and exams to draw from. One would think that with all of these prepared materials, this course could become intellectually stagnant, but I have found it to be just the opposite. Because it is an introductory course, each new semester brings a new group of students with varied backgrounds, diverse interests, and new and interesting questions, and this past semester was no different.

I was at the point in the semester where the class was investigating position, velocity, and acceleration. I discussed the connection between the average rate of change over an interval and the instantaneous rate of change at a point in that interval, and in particular, I focused on the relationship between average velocity and instantaneous velocity to motivate the definition of the derivative. To emphasize this idea, I included on their exam a question involving average velocity. I gave the students a quadratic function and asked them to calculate the average velocity over a given interval. Pretty straightforward, I thought, but when I returned their exams, one of my students asked why he had lost points on this problem. It was true that he got the final numerical answer correct, but he hadn't used the average velocity formula he had learned in our course. Instead of calculating the average velocity over the given interval, he had calculated the average of the velocities at the end points of the given interval. When I explained this to him, he stated that he didn't understand the difference because he had learned the latter formula to calculate average velocity in his physics class.

Given a position function of an object, a calculus text defines average velocity over a closed time interval to be the displacement of the object over the given time interval divided by the time elapsed. Why would physics teachers be teaching their students that the average velocity over an interval is the average of the velocities at the endpoints of the given time interval? Clearly, this is not true in general. Take for example a driver who leaves home at 8 a.m. and arrives at a friend's home 50 miles away at 10 a.m. The average velocity the driver was traveling is 25 miles per hour, whereas the average of the velocities at 8 a.m. and 10 a.m. is 0 miles per hour.

To get to the bottom of my inquiry, I posed the same question to a couple of my colleagues who were also teaching single-variable calculus. The first thing we realized was that the reason why my student got the final numerical answer correct was because I had given them a quadratic function, and it can be easily shown that if a particle has a quadratic position function, that is, has constant acceleration, the average velocity over an interval is the average of the velocities at the endpoints of the given time interval. Here's a quick proof.

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Theorem. Suppose a particle in motion has position function $f(t) = at^2 + bt + c$, that is, constant acceleration over the interval $[x_0, x_1]$ for real values *a, b,* and *c*. Then by definition the average velocity is equal to

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{a(x_1)^2 + bx_1 + c - a(x_0)^2 - bx_0 - c}{x_1 - x_0} =$$
$$\frac{a(x_1 - x_0)(x_1 + x_0) + b(x_1 - x_0)}{x_1 - x_0} = a(x_1 + x_0) + b.$$

Similarly, the average of the velocities at $t = x_0$ and $t = x_1$ is equal to

$$\frac{f'(x_0) + f'(x_1)}{2} = \frac{(2ax_0 + b) + (2ax_1 + b)}{2} = a(x_1 + x_0) + b.$$

But the question still remained, why would physics teachers be teaching their students that the average velocity over an interval is the average of the velocities at the endpoints of the given time interval? As it turns out, they were not, at least not in general. However, constant acceleration is a common theme in physics that occurs in many physical systems in nature. Falling bodies (sans air resistance), bodies sliding on an incline or rough surface, or charged particles moving in a constant electric field, all can be modeled as experiencing constant acceleration.

So it follows from above that the average velocity over an interval is equal to the average of the velocities at the endpoints of the given interval in these cases.

As it turns out, many introductory physics texts, both at the high school and college level, break out this special case of straight-line motion into specific sections in their exposition and include formulas for students to apply in these cases. Because many students who take physics in high school have not yet seen any calculus concepts, the equations they are presented with are not explicitly derived.

In addition, it is common for college students to take introductory calculus and physics either concurrently or one after another, so it is understandable that a student would work to unify the concepts in these two subjects. Thus, it is easy for either population of students to try to use these formulas in their introductory calculus class, unaware that they are assuming a special case, if the subtleties are not explicitly pointed out.

Handling Such Subtleties

So what do we as mathematics professors do? We, like our students, also have varied backgrounds and diverse interests. Some of us have never taken a physics course, some of us were mathematics and chemistry double majors, and some of us became mathematics majors simply because of the beauty of pure mathematics.

Whatever our background, we are mathematics professors now and situations like the one I found myself in will occasionally come up. Does this mean we need to continually consult with every department that utilizes calculus and make sure we understand the "subtleties" of their subject? Definitely not, but with so many of our first-year students taking common courses in mathematics, chemistry, physics, and economics, we should be aware that special cases will arise and be willing to work through the details with interested students when they do.

I'm glad I was. My student and I both learned from this experience. He gave me the opportunity to look at a familiar topic with the eye of a physicist, and I taught him the importance of context when using a formula. Specific adventures such as the one my student and I encountered will undoubtedly strengthen my approach to teaching this course and my students' ability to think like mathematicians.

P.S. Because my student's formula was technically correct, I gave him some of his points back, but I told him that in the future, if he uses a formula that we didn't learn in class, he'll need to prove why his formula was valid. He understood, and he still stops by every now and then to make sure the formulas he is taught in other classes are true in general.

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