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On the Geometry of Linking Production to Cost: The Case for Cobb-Douglas

Stephen E. Erfle
International Business and Management
Dickinson College, erfle@dickinson.edu

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Stephen E. Erfle
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Dickinson College
erfle@dickinson.edu

Abstract

In introductory and intermediate microeconomics courses, students are taught about production functions and cost functions but rarely are they shown a direct link between production and cost. If we assume production is Cobb-Douglas, then the *CDProductionToCost.xlsx* Excel file shows the link between short-run, SR, and long-run, LR, production and cost in geometric detail. This geometric approach is particularly useful in classes without a calculus prerequisite. The file has three interactive figure sheets and four information sheets. The figure sheets do not require a knowledge of Excel to operate; they can be manipulated simply using sliders and click-boxes. To focus attention on the graphical material presented, the equations that created each figure are hidden and the interactive figure sheets are write-protected. The information sheets are not write-protected so instructors can alter them as they wish. The accompanying paper provides additional graphical and algebraic detail and suggests alternative ways to use the file to deliver material to students. Although the main purpose of this file is to show the geometric link between production and cost, the supplementary sheets allow instructors (and students) to explore the underlying production function and cost-minimizing solutions that allow one to link production to cost.

Keywords: Producer Theory, Cobb-Douglas Production, Cost Minimization, Cost Functions, Power Functions

1. Introduction

In the theory of the firm discussion in microeconomics courses, students learn about production functions and cost functions. Production and cost are interrelated concepts but the direct linkage between them is rarely discussed in textbook expositions of these topics. This paper discusses that linkage in general, and for the specific case when production is Cobb-Douglas.

The simplest graphical connection between production and cost is to use the SR production function (most commonly called a total product curve, TP(L)) and point out that if you multiply labour values on the horizontal axis by w, the price of labour, and flip axes (reflect, actually) then you end up with variable cost, VC(Q). A particularly elegant reflected geometric version is presented in [1]. Figure 1 is a modified version of this figure, based on w = \$30. The right half shows the typical shape of a TP(L) curve which is first convex up due to initial increasing marginal productivity of labour, but then is convex down due to the law of diminishing returns. The left half is linked to the right by a common vertical axis, Q, the dependent variable in TP(L) and the independent variable in VC(Q). It shows VC(Q) once Figure 1 is rotated clockwise 90°. Marginal product of labour, MP_L, the slope of TP(L), for the first five units of labour is 1, 2, 3, 2, 1 using the right half of Figure 1. Marginal cost, MC(Q), the slope of the VC(Q) curve in the left half (the top half once Figure 1 is rotated 90°), is readily seen as \$30 for the first unit of output, \$15 for the second and third unit, \$10 for the next three units, \$15 for the seventh and eighth units, and \$30 for the ninth unit of output. Quite simply, variable cost is a reflected version of SR production, scaled by the wage rate. One can quickly drive home this point by noting that if we had assumed w = \$3 rather than w = \$30, the only difference would be that all axis labels on the "negative x axis" would now be a power of 10 smaller. Importantly, the shape of VC(Q) would not change due to this change in wage rate.

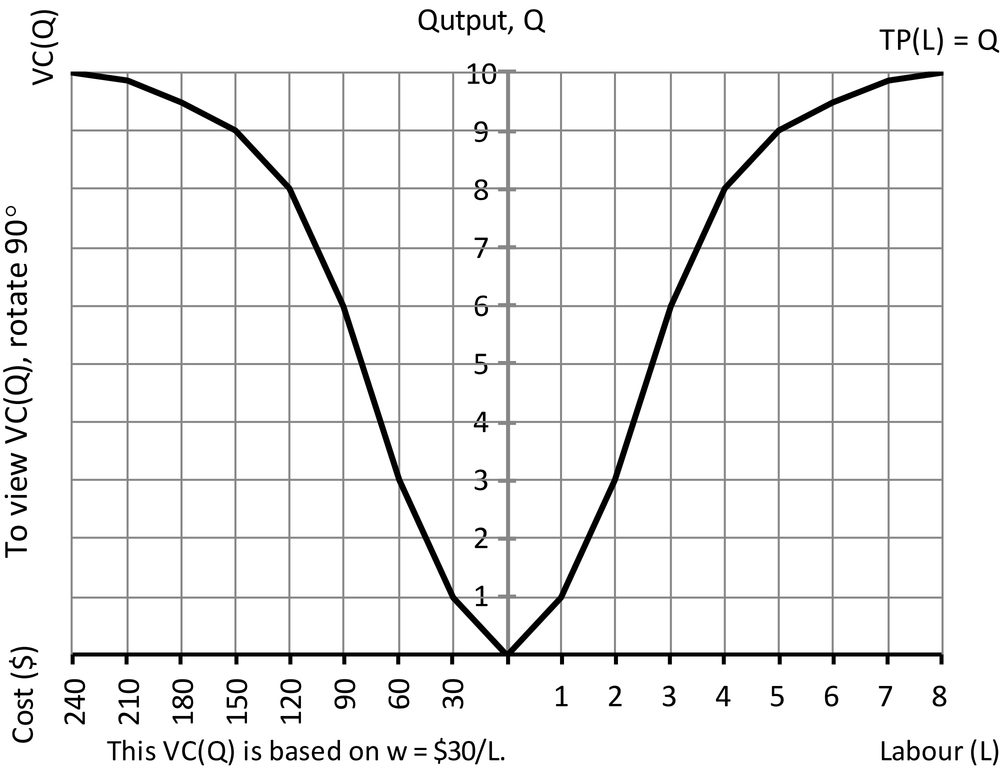

Figure 1: Short-run Production, TP(L), and Variable Cost, VC(Q)

Given VC and fixed cost, one obtains SR total cost, STC, and SR average variable and total cost curves, SAVC and SATC. Different levels of capital would produce different TP(L) curves and total and per-unit cost counterparts. From these SR cost curves, we could, in theory, obtain LR cost curves since LR total cost, LTC, and LR average cost, LAC, are the envelopes of STC and SATC, respectively.

Using a Cobb-Douglas production function, CD, we can move beyond this theoretical linkage. We can derive SR and LR total and per-unit cost curves and see how those curves adjust to changes in the parameters of the CD production function. We can also see how these cost curves adjust to changes in the price of labour (often called w, the wage rage) and the price of capital (often called r, the rental rate on capital). The *CDProductionToCost.xlsx* Excel file shows in geometric detail the relation between production and cost in both the SR and LR.

The second section of this paper describes strategies for using this file as a teaching tool in the classroom. The third section provides the mathematical detail behind these figures.

2. Using *CDProductionToCost.xlsx* in the classroom and for assessment

The typical discussion of theory of the firm proceeds from SR production with a single variable input (like labour in the right half of Figure 1 above), to production with multiple variable inputs, to the cost-minimizing method of producing a given level of output in the SR and in the LR, to SR and LR cost functions.

Often producer theory occurs after consumer theory is discussed so that producer theory is built on a consumer theory foundation. Production functions are like utility functions with some notable differences. Production is cardinal while utility is ordinal. Examples of specific utility functions encountered in consumer theory such as perfect substitutes, perfect complements and CD re-emerge on the producer side. Isoquants are like indifference curves but once again, the numbers mean something cardinal in the producer theory context. MP is like marginal utility and the ratio of marginal products, the slope of the isoquant (called variously the marginal rate of technical substitution, MRTS, or MRS, or RTS), is analogous to MRS on the consumer side. An isocost line is like a budget constraint on the consumer side, but the consumer faces a single budget constraint while the firm faces multiple isocost lines. The main consumer optimization problem requires that a consumer **maximizes** utility subject to a budget constraint. The producer side analog is that a producer **minimizes** the cost of producing a given level of output subject to a production constraint. Both solutions require tangency (between indifference curve and budget constraint, MRS = P_x/P_y, or isoquant and isocost, MRTS = w/r). Formally, the two problems are duals of one another.

2.1. Linking CD production to CD cost

The main purpose of the *CDProductionToCost.xlsx* Excel file is to show the link between production and cost using one of the most commonly used production functions in microeconomics, CD. This occurs on the *CDCosts* sheet, a screenshot of which is shown as Figure 2.

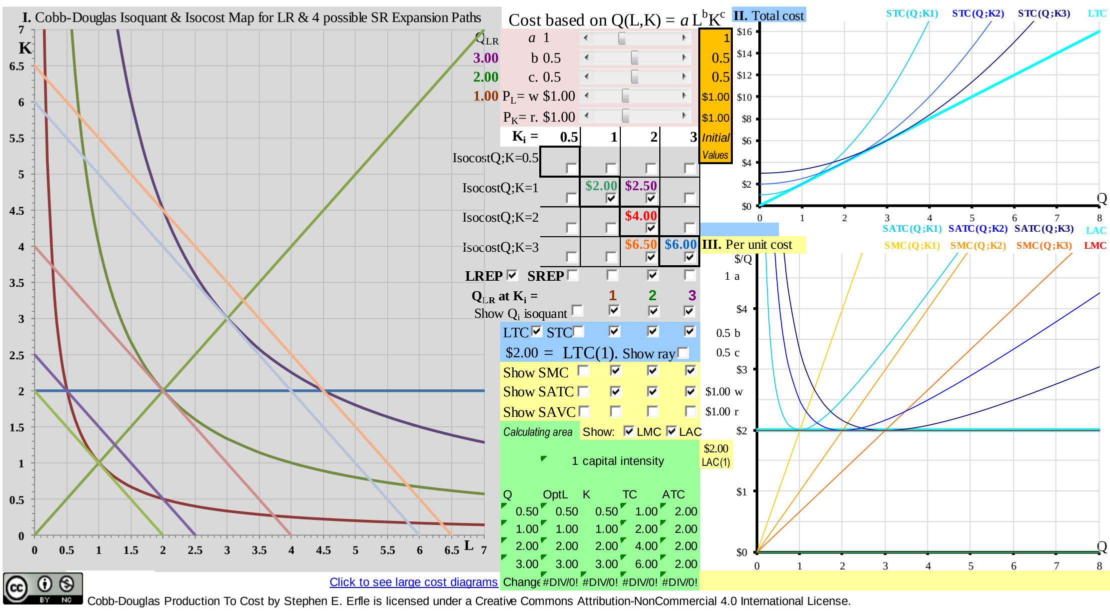

Figure 2: Screenshot of the *CDCosts* sheet showing the baseline scenario, Q = $(K{\cdot}L)^{0.5}$ with w = r = $1

The *CDCosts* sheet includes three interrelated interactive graphs with click-boxes color-coded to each graph. I (gray) is an isoquant map allowing a discussion of SR production (with K fixed) versus LR production, II (blue) shows STC and LTC, and III (yellow) shows SR and LR per-unit costs.

Sliders on the rose-colored background on the *CDCosts* sheet allow users to adjust the three parameters of the CD production function in Equation 1, *a*, b, and c, together with factor prices w and r and follow what happens to SR and LR costs. MRTS for CD production is shown as Equation 2. I use this sheet to wrap-up the discussion of production and cost by providing students with a visual display of how the various total and per-unit cost curves change as each of the underlying parameters change.

$$Q(L, K) = aL^bK^c \tag{1}$$

$$MRTS = MP_L/MP_K = (abL^{b-1}K^c)/(acL^bK^{c-1}) = (b/c)\cdot(K/L) \tag{2}$$

The five initial parameter values, noted in orange in Figure 2, are the simplest version of CD production, $Q(L, K) = (LK)^{0.5}$, together with the simplest assumption regarding factor prices, w = r = \$1. Since b+c = 1, we have constant returns to scale, CRTS, and hence constant LR per-unit costs. (If you have discussed homogeneity in class, note that CD is homogenous of degree b+c, so this example is CRTS because b+c = 1.) Because b = c and w = r, the MRTS = w/r tangency occurs when L = K so the LR expansion path, LREP, is L = K in panel I of Figure 2. The point (1, 1) in panel I produces 1 unit of output at minimum cost of \$2 in all three panels. In panel I, this is the tangency of isoquant and isocost at (1, 1), in panels II and III, it is seen as the point (1, \$2). Because this production process is CRTS, LAC = LMC = \$2 in III. This is equal-weighted CRTS, CD production that produces output at minimum ATC of \$2/unit.

2.1.1. Comparative statics analysis

From here, I ask: How does each parameter alter production and cost? For example, if the scaling factor *a* increases from 1 to 2, output for any bundle of inputs doubles. Put another way, the cost of producing a given level of output is cut in half. Check that this is true by moving the *a* slider all the way to the right. After you examine one parameter, return it to its initial value and turn to the next parameter. I suggest discussing w and r before attacking b or c.

An increase in w increases VC but not fixed cost in the SR. See this by pointing out to students that the fixed cost points on the vertical axis of panel II do not change as w changes. Instead, increasing w appears to move STC curves upward (because SVC moves upward). With b = 0.5, SVC and hence STC curves are quadratic so that SAVC and SMC are linear. (The equations producing the graphical results in Section 2 are less important than are the graphical results themselves. These equations are discussed in Section 3 of this paper.) With b+c = 1, we have CRTS as noted above, so that LTC is linear, and LAC = LMC = slope of LTC are flat. An increase in w makes each SMC steeper and appears to float minimum SATCs upward (minimum SATC is the point where SMC = SATC).

The reason SVC is quadratic, and how it moves as w changes, is most readily understood by seeing where SVC comes from in the CD case, or to put it in terms of the above discussion, what Figure 1 looks like for SR CD production functions. This is done in two steps. First, it is worth showing a more complete isoquant map for CD production in order to consider in greater detail the SR production function in this instance, $TP(L) = K_0^{0.5} \cdot L^{0.5}$. Second, show how these TP(L) curves are linked to VC(Q).

The TP(L) function depends critically on the SR capital stock, K_0. The *CDCosts* sheet examines four capital stock levels, $K_0 = 0.5$, 1, 2, and 3, but the SR with $K_0 = 2$ is the only one shown in panel I of Figure 2 (as the SR expansion path, SREP, blue horizontal line at K = 2 in that panel). You can also see isoquants on the LREP associated with each of these capital stocks, three of which are shown in panel 1 of Figure 2. A more complete isoquant map for CD production is provided in the *CDProduction* sheet, a screenshot of which is shown as Figure 3.

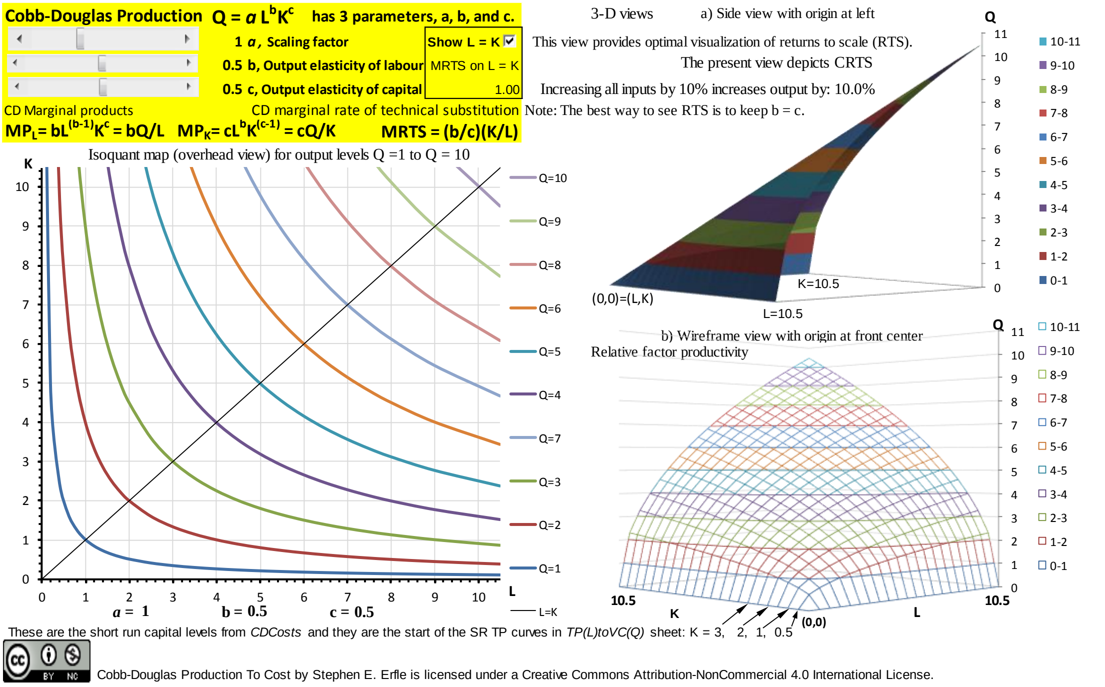

Figure 3: Screenshot of the *CDProduction* sheet

This sheet allows you to adjust the three parameters of the CD production function via sliders. Since production does not depend on factor prices, no sliders are present for w and r. This sheet shows CD production from three perspectives. The overhead view is simply a more detailed isoquant map showing fixed output levels from 1 to 10. MRTS = 1 on the main diagonal as shown because this is equal weighted CD. The CRTS nature of this scenario is seen as isoquants equidistant from one another (and by having the 3-D side view outline be linear). The 3-D wireframe view shows the starting point of four TP(L) curves (SR CD production functions). These curves are the centrepiece of the *TP(L)toVC(Q)* sheet that links SR production to VC. Figure 4 shows a screenshot of the *TP(L)toVC(Q)* sheet. Each is a power function of power b. In Figures 3 and 4, each is a square root function, given b = 0.5.

The TP(L) curves in the left panel in Figure 4 are the CD production analogs of the general TP(L) curve shown in the right half of Figure 1. (Note that, unlike in Figure 1, these TP(L) curves have no range with increasing MP_L.) Each TP curve is a power function of power b = 0.5. Higher output is produced for a given level of L the larger is K. The SVC curves in the right panel of Figure 4 are analogous to VC curve in the left half of Figure 1 (once Figure 1 is rotated 90° clockwise). Note that the VC curves are power functions of power 1/b = 2 with higher levels of K (fixed cost) leading to lower VC curves. Once again, you can adjust the three CD parameters, but the most instructive thing to do initially is to simply ask students: What happens to TP(L) and VC(Q) if w increases? Nothing happens to SR production (in the left panel), but VC increases (in the right panel) as w increases. If w doubles (from \$1 to \$2), then VC(Q) doubles for each of the three SVC curves. Use the w slider to see that this is indeed the case (note, for example that the red VC(Q; K=1) curve contains the point (5, \$25) in the right panel when w = \$1 (because $Q(25,1) = (25\cdot 1)^{0.5} = 5$), but (5, \$50) when w = \$2).

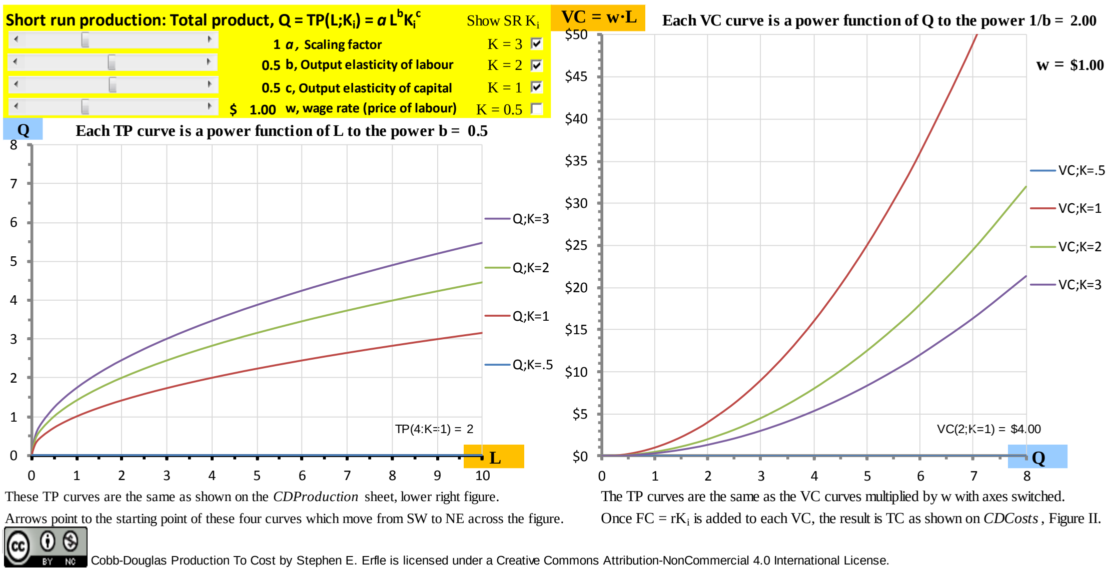

Figure 4: Screenshot of the *TP(L)toVC(Q)* sheet

Returning to the *CDCosts* sheet, note that increasing w makes the isocost lines in panel I steeper because the slope of isocost is w/r (actually -w/r but typically we strip the minus sign when discussing slope of isocost, so we think of it as the market determined trade-off between capital and labour). When w = \$2, the LREP is K = 2L. Note in this instance that LAC increases to \$2.83, rather than \$3. It is instructive to ask: Why this is the case? The answer is that cost would increase to \$3 only if there were no substitution towards the input that has become less expensive on a relative basis, K in this instance. (The input bundles (1, 1) and $(2^{-0.5}, 2^{0.5})$ both produce Q = 1 unit of output but the former costs \$3 and the latter costs \$2.83.)

Return w to \$1 and consider what should happen when r increases. An increase in the cost of capital has no effect on VC but increases fixed cost = r·K. Use the r slider to show that as r increases, each SMC in panel III does not change and the STC curves in panel II do not change shape, they simply shift up as fixed cost increases. If r increases from \$1 to \$2, the fixed cost points in panel II of Figure 2 increases from \$1, \$2 and \$3 to \$2, \$4, and \$6. Note also that in this instance, w/r decreases from 1 to ½ and the LREP is K = ½·L in panel I and LAC increases to \$2.83 in panel III because of substitution in the direction of the good that has become less expensive on a relative basis, L in this instance.

Return r to \$1 and consider what should happen when the output elasticity of labour, b, changes. For fixed c = 0.5, if b increases from 0.5, then b+c > 1 and the production process exhibits increasing returns to scale, IRTS. If b decreases from 0.5, then b+c < 1 and the production process exhibits decreasing returns to scale, DRTS. An IRTS scenario is shown in Figure 5 with b = 0.6. Several points can be made using this figure.

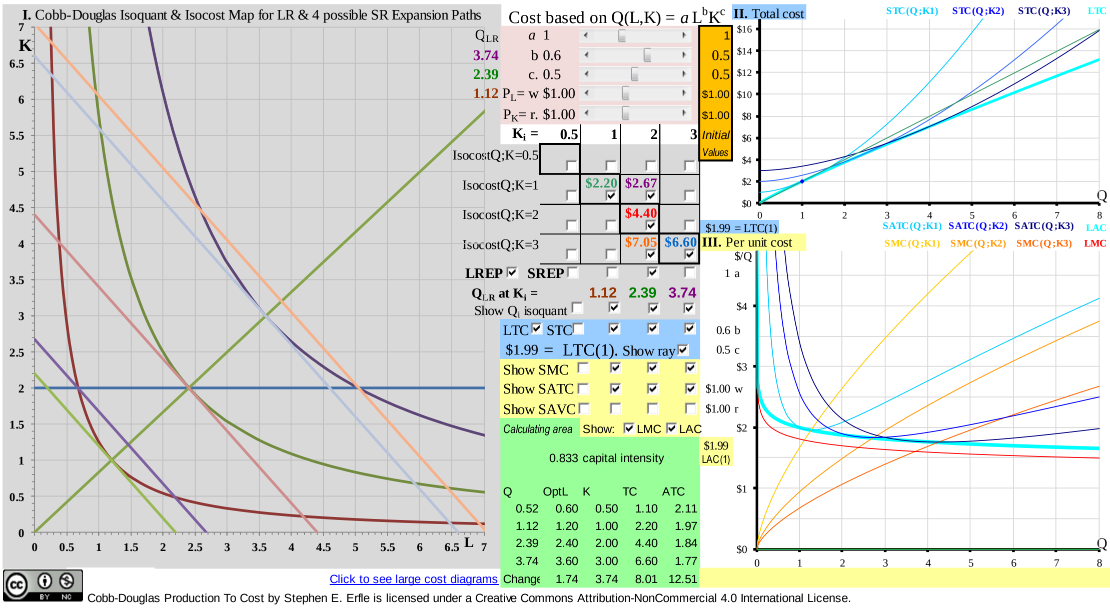

Figure 5: Screenshot of the *CDCosts* sheet showing an IRTS scenario

The LREP in Figure 5 is K = 5/6·L (because MRTS = (0.6/0.5)·(K/L) and w = r). Output level of each isoquant on the *CDCosts* sheet varies in panel I so that the cost-minimizing bundles occur at one of four K levels, three of which are shown in Figure 5 (K = 1, 2, and 3). The level of these outputs is noted in the upper right corner of the panel. The IRTS nature of production can be seen in each of the three panels of Figure 5 because inputs double from (1.2, 1) to (2.4, 2) and triples to (3.6, 3) on the LREP but output more than doubles from 1.12 to 2.39 and more than triples to 3.74. (Because production is homogeneous of degree b+c = 1.1, we know that 2.39 = $2^{1.1}$·1.12 and 3.74 = $3^{1.1}$·1.12.) To see this IRTS in production more directly, go to the *CDProduction* sheet and use the b slider until b = 0.5. IRTS is seen in the Isoquant map by the isoquants become closer and closer together as Q increases and because the 3-D side view outline appears convex rather than linear. IRTS in production has implications for cost.

Because production in Figure 5 has IRTS, LR total costs increase less than linearly in panel II and LR per-unit costs are a declining function of output in panel III. The latter is readily visible, but the former can be seen using the LTC(1) ray in panel II (which can be added or removed using the click-box in cell K21 of the *CDCosts* sheet). That ray appears to go through the point (7, $14) but LTC(7) = $11.68, less than seven times LTC(1) = $1.99. (Indeed, because production is homogeneous of degree b+c = 1.1, LTC is homogeneous of degree 1/(b+c) = 1/1.1 = 10/11 and LAC and LMC are homogeneous of degree 1/(b+c) – 1 = -1/11.)

An increase in b to 0.6 has implications for SR costs as well. Because the SR production function is a power function of degree 0.6, VC is a power function of power 1/0.6 = 5/3. This can readily be shown using the *TP(L)toVC(Q)* sheet by adjusting the b slider so that b = 0.6. SVC and STC functions become closer to linear but remain convex upward (as will always be the case if b < 1, a condition required in order to have declining MP_L).

When b > 0.5, SMC and SVC are both power functions of degree 1/b – 1 < 1. If b = 0.6, SMC and SAVC are power functions of degree 2/3. More generally, SMC and SAVC are convex downward if b > 0.5, linear if b = 0.5, and convex upward when b < 0.5.

Finally, return b to 0.5 and consider what should happen when the output elasticity of capital, c, changes. For fixed b = 0.5, if c increases from 0.5, then b+c > 1 and the production process exhibits IRTS. This has similar LR implications for LTC, LMC, and LAC to the change in b just discussed. However, STC and SVC remain quadratic and SMC and SAVC remain linear. As c increases, less labour is required to produce a given level of output for fixed capital and hence the cost of producing a given level of output declines. This is visible as SMC curves rotating outward as c increases.

Table 1 summarizes the results of the comparative statics analysis discussed above. Two versions of this table are provided on the *Handout* sheet depending on whether the instructor wishes to provide a completed table or wants to use this as an exercise.

Table 1: A completed comparative statics table

Cost curves based on the Cobb-Douglas production function Q(L,K) = $a\,L^bK^c$

<table>
<tr><td colspan="5">Shape and orientation of long run Cobb-Douglas cost curves</td></tr>
<tr><td>Condition</td><td>RTS</td><td>Shape of LTC</td><td colspan="2">LAC and LMC</td></tr>
<tr><td>b+c < 1</td><td>DRTS</td><td>convex upward</td><td colspan="2">increasing in Q, LMC > LAC</td></tr>
<tr><td>b+c = 1</td><td>CRTS</td><td>linear</td><td colspan="2">constant in Q, LMC = LAC</td></tr>
<tr><td>b+c > 1</td><td>IRTS</td><td>convex downward</td><td colspan="2">decreasing in Q, LMC < LAC</td></tr>
<tr><td colspan="5">Shape and orientation of short run Cobb-Douglas cost curves</td></tr>
<tr><td>Condition</td><td colspan="2">Shape of STC (and SVC not shown)</td><td>Shape of SMC and SAVC</td><td>Orientation</td></tr>
<tr><td>0 < b < 0.5</td><td colspan="2">Power function greater than quadratic</td><td>More than linear in Q</td><td rowspan="3">SMC > SAVC</td></tr>
<tr><td>b = 0.5</td><td colspan="2">Quadratic cost function</td><td>Linear in Q</td></tr>
<tr><td>0.5 < b < 1</td><td colspan="2">Power function less than quadratic</td><td>Less than linear in Q</td></tr>
<tr><td>Change parameter</td><td colspan="4">Effect of an increase in this parameter.</td></tr>
<tr><td>Scaling factor, a</td><td colspan="4">Decreases all short and long run cost curves.</td></tr>
<tr><td>Wage rate, w</td><td colspan="4">Increases STC, SAVC, SATC, SMC and all long run cost curves.
Does not change SFC or SAFC (both not shown).</td></tr>
<tr><td>Capital cost, r</td><td colspan="4">Increases SFC (not shown) therefore STC, and SATC and all long run cost curves.
Does not change SVC, SAVC or SMC.</td></tr>
</table>

2.2. Discussing CD Production

The initial discussion of production functions often introduces CD production as a middle-ground between no input substitutability Leontief production and perfect substitute production. The rate at which that substitution occurs is not constant as you move along the isoquant (as is the case for perfect substitutes). Capital is more readily substituted for labour if the input bundle is capital intensive to begin with, *ceteris paribus* – this means that the MRTS is larger when K/L is larger. Conversely, labour is more readily substituted for capital if the input bundle is more labour intensive to begin with, *ceteris paribus* – this means that the MRTS is smaller as K/L is smaller. Put another way, MRTS declines as we “walk” down an isoquant – isoquants are convex. The degree of substitutability is determined by the technological opportunities available for producing the specific good in question. A function that provides a middle-ground level of flexibility is the CD production in Equation 1. Note that MRTS declines as we walk down the isoquant according to Equation 2. The *CDProduction* sheet can be used to explore CD production much as CD cost was examined in Section 2.1. One can readily see how production changes as each of the three CD parameters change. Some of this discussion replicates what was done earlier for CD cost, but other parts are best approached in a different fashion when we focus on production. As with the CD cost analysis, it is worthwhile to start from the simplest CD form possible, equal-weighted CRTS CD production that produces 1 unit of output when L = K = 1, $Q(L, K) = (L\cdot K)^{0.5}$. This is the scenario shown in Figure 3.

The effect of changing the scaling factor *a* is easiest to see using the isoquant map in *CDProduction* sheet. With *a* = 1, the 10 isoquants associated with producing 10 integer output levels from 1 – 10 pass through (1, 1), (2, 2), …, (10, 10). If *a* increases, more can be produced for a given bundle of input, or, the same amount can be produced from a smaller amount of inputs. Verify that when *a* = 2, the same 10 isoquants pass through (0.5, 0.5), (1, 1), …, (5, 5) using the *a* slider on the *CDProduction* sheet. Once done, you should return the *a* slider to 1.

The easiest way to examine relative factor productivity is to maintain CRTS production by varying the output elasticities for both factors. This is readily accomplished by increasing one factor as much as the other factor decreases, to keep b+c = 1, or, put another way, c = 1-b. MRTS in this instance simplifies to MRTS = (b/(1-b))·(K/L). Along the L = K diagonal, this simplifies to MRTS = b/(1-b). When b = 0.6, for example, the 10 isoquants pass through (1, 1), (2, 2), …, (10, 10) but now have a slope of 0.6/0.4 = 1.5 along the main diagonal. Labour is now relatively more productive and the isoquant map tilts in the labour direction in both the isoquant and 3-D wireframe view. The situation, of course, reverses if b < 0.5 (because then, c > 50%).

Relative factor productivity filters through to cost-minimizing input choice as well. To examine this issue, return to the *CDCosts* sheet and set b = 0.6 and c = 0.4. To reduce sensory overload, it is worthwhile to remove all cost functions and show only one isoquant. The result is shown in Figure 6.

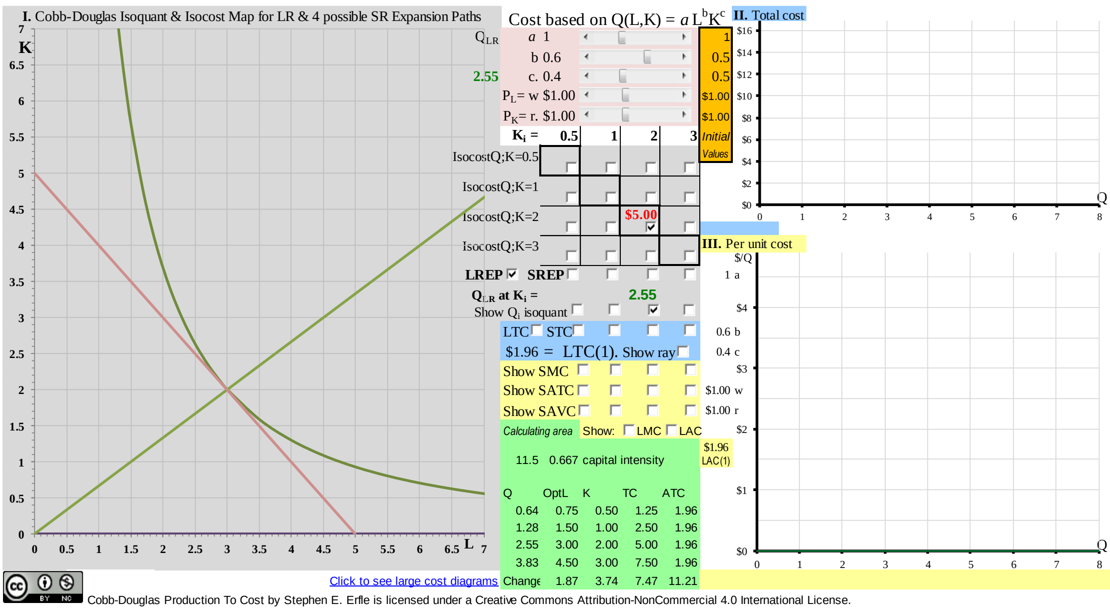

Figure 6: Screenshot of the *CDCosts* sheet showing cost-minimizing input choice

Note that 60% of cost is devoted to labour (3/5) and 40% (2/5) is devoted to capital, a ratio that is determined by the relative size of the CD exponents. (This works when $b+c \neq 1$ as well. For example, $b = 0.75$ and $c = 0.5$ is IRTS, and $b = 0.45$ and $c = 0.3$ is DRTS but both have $b/(b+c) = 0.6$ so 60% of cost is devoted to labour in each instance and $c/(b+c) = 40\%$ is devoted to capital.) This ratio does NOT depend on factor prices as is easy to verify by changing w and r. As either factor price changes, the isocost line rotates on (0, 5) because the isoquant chosen is optimal given K = 2. One can readily verify that this works for other relative factor ratios (for example if b = 0.3 and c = 0.6 then 1/3 of cost is devoted to labour and 2/3 is devoted to capital).

The easiest way to see returns to scale is to vary b and c while maintaining b = c. When b = c < 0.5, production exhibits DRTS and isoquants on the isoquant map panel in *CDProduction* sheet become farther and farther apart as output increases but maintain MRTS = 1 on the main diagonal (because b = c). The 3-D side view is convex down with DRTS. Conversely, when b = c > 0.5, production has IRTS. Isoquants get closer together and the 3-D side view is convex upward.

2.3. Discussing cost minimization in the SR and LR

The discussion in Section 2.2 has assumed that the firm has complete freedom in determining how much of each factor to use in production. One of the hallmarks of production processes is that some aspects of a production process are easier to adjust than are others. Over the short term, plant, property and equipment are largely fixed while labour and materials can more readily be adjusted to produce more or less output. For example, a firm can use existing machinery but have the workers work longer hours or put on a second shift if they wish to increase output. Conversely, the firm can move to a restricted work week if a decrease in output is required. The firm finds it impractical, or even impossible, to quickly expand or contract property, plant and machinery as a way of adjusting output in the SR. We typically model the SR/LR distinction with only two inputs. Economists model this differential ability in an (L, K) model by saying that in the SR, capital is fixed. This distinction can be examined using panel I of the *CDCosts* sheet using the same equal-weighted CRTS CD production used above, $Q(L, K) = (L\cdot K)^{0.5}$. Figure 7 shows the setup for this discussion (which occurs prior to discussing cost curves, as a result they are removed).

To move from a LR analysis to a SR analysis we must know how much of the fixed factor of production the firm has available to it in the short run. One way to determine this is to examine the firm's expectations regarding future production (another way to proceed is to simply assume a given level of capital stock, K_0, in the short run). Suppose the firm expects to produce 2 units of output per unit of time and they do not expect relative input prices to change over time. Cost-minimizing production of 2 units of output occurs at input bundle (2, 2) in Figure 7. Therefore, the firm invests in 2 units of capital. This investment decision is a LR decision. Once this decision is made, the firm is now embedded in a SR meaning that changes in output can only occur by adjusting labour. The horizontal line at $K_0 = 2$ in panel I is the SREP, this line is an overhead view of the SR production function $TP(L) = (2L)^{0.5}$. In this short run, if you want to produce Q = 3, you must use 4.5 units of labour ($3 = (2\cdot 4.5)^{0.5}$), and if you want to produce Q = 1, you can do so with 0.5 units of labour ($1 = (2 \cdot 0.5)^{0.5}$). With w = r = \$1, costs are simply STC = 2 + L so, for the output levels under discussion, STC(1) = \$2.50, STC(2) = \$4, and STC(3) = \$6.50. These are the three isocost lines passing through the points (0.5, 2), (2, 2), and (4.5, 2). Only the middle one is cost-minimizing in the LR sense meaning that MRTS = w/r. But once we have chosen our plant size (that is, once we have decided to invest in K_0 = 2 units of capital), this equality is IRRELEVANT.

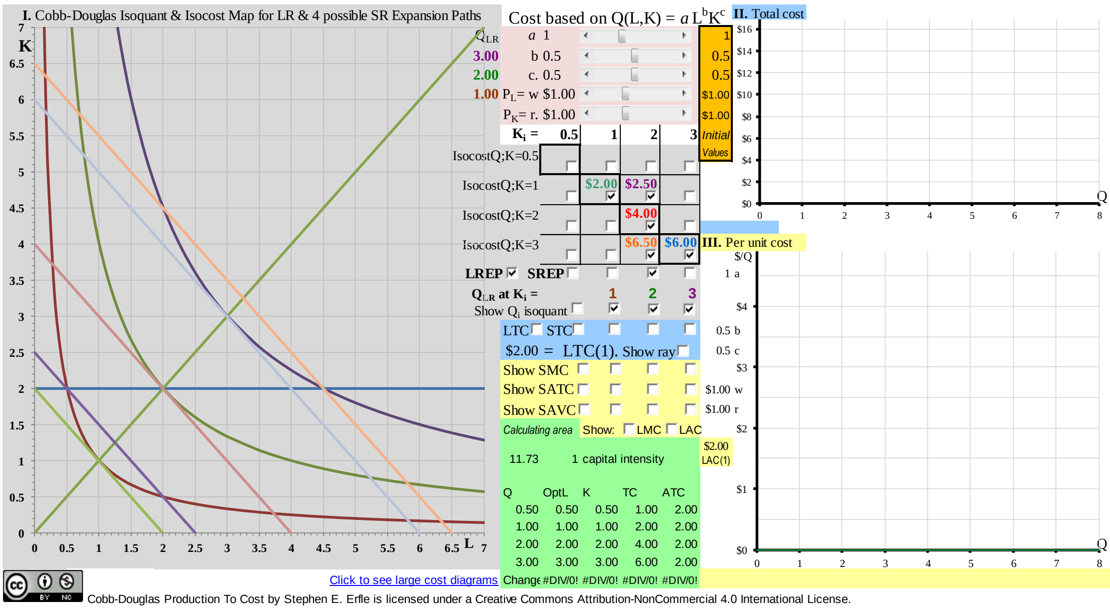

Figure 7: Screenshot of the *CDCosts* sheet for discussion of SR versus LR cost minimization

There is a cost to this restriction (beyond the cost of the inputs themselves); SR costs are always as high as or higher than LR costs. In Figure 7 we see that producing Q = 3 in the SR costs an extra \$0.50 relative to using the (3, 3) input bundle on the LREP. Similarly, producing Q = 1 in the SR costs an extra \$0.50 relative to using the (1, 1) input bundle on the LREP.

While a SR expansion in output is reasonably straightforward to conceptualize, students often have trouble with SR contractions in output. It is easy to understand that a plant may become too small if you suddenly wish to produce more than you have typically produced. The only solution is to increase labour since capital cannot increase in the SR. However, why not just leave some of your plant unused if you wish to contract output in the SR? Students often examine Figure 7 and suggest the following analysis:

> "2 units of capital are available in the SR, but cost-minimizing production only requires the use of 1 unit of capital and 1 unit of labour at a cost of \$2. Therefore, I will only use 1 unit of capital."

The problem with this analysis is that it ignores the fixed nature of capital in the short run. If you follow this strategy, you still must pay for the 1 unit of capital that is left unused in the short run (so STC would be \$3 rather than \$2). This unit of capital is part of the firm's SR capital stock and the firm must pay rent on it regardless of usage in the SR.

Three text sheets round out the *CDProductionToCost.xlsx* file. These sheets are not write-protected so instructors can alter them as they wish. The *InstructorNotes* sheet prints as seven pages and provides additional details for how to use the file for classroom and homework purposes. It also discusses the equations behind the CD cost figures (as does Section 3 below). The *StudentQuestionsAndNotes* sheet prints as six pages and provides students 41 questions in three topic areas; CD production, CD cost-minimization, and CD costs, as well as notes which help students describe how cost curves change when a, b, c, w, or r change. These notes focus on how to describe a curve by its shape and placement. Students are also provided with the CD cost equations, but they can gain substantial insights from the geometry of these curves without ever being exposed to the equations that created them. The three-page *Answers* sheet should be deleted prior to student distribution.

3. Mathematical detail

Some intermediate level microeconomics texts discuss the algebra behind specific cases of CD cost. Most do so with without geometric support and at least one does so with warnings such as the one presented in [2]. An elegant derivation of LTC for CD production is provided in [3]. The discussion of SR and LR CD cost curves presented here follows [4]. That exposition argues that the algebra involved is more effectively examined by showing the essential power function structure of the algebra in geometric relief.

3.1. CD SR costs

Start from CD production in Equation 1 setting $K = K_0$ we have SR CD production as a function of L (as noted above, this is often called a total product curve):

$$Q(L; K_0) = aL^b(K_0)^c \tag{3}$$

(K_0 after the semicolon implies that K is parametrically given.) Invert Equation 3 by solving for L to obtain the labour required to the produce Q units of output in the SR:

$$L = ((Q/a)(K_0)^{-c})^{(1/b)} \tag{4}$$

Multiplying Equation 4 by w we obtain SVC(Q) just as we did in Figure 1 for a general function, and in Figure 4 for a specific version of CD production:

$$SVC(Q) = w\cdot((Q/a)(K_0)^{-c})^{(1/b)} = Q^{(1/b)}\cdot\mathbf{[}w\cdot(K_0^{-c}/a)^{(1/b)}\mathbf{]} = \mathbf{Q^{(1/b)}}\cdot G(w, a, b, c, K_0) \tag{5}$$

The final version of Equation 5 (and subsequent equations) shows the power function structure of SVC(Q). The bracketed term, $G = G(w, a, b, c, K_0)$, acts as a scaling factor for the power function. Note that even when written in shorthand as G, this scaling factor is a function of SR capital stock, K_0, w, and the three CD parameters, *a*, b, and c. Adding fixed cost, $F = rK_0$, we obtain STC(Q), which is a vertical translate of a power function:

$$STC(Q) = rK_0 + Q^{(1/b)}\cdot\mathbf{[}w\cdot(K_0^{-c}/a)^{(1/b)}\mathbf{]} = F + \mathbf{Q^{(1/b)}}\cdot G \tag{6}$$

Per-unit costs are readily obtained from Equations 5 and 6. Average cost functions are obtained by dividing each by Q and marginal cost is the derivative of SVC (or STC):

$$\text{SAVC}(Q) = \mathbf{Q}^{(1/b-1)} \cdot G \tag{7}$$

$$\text{SATC}(Q) = F/Q + \mathbf{Q}^{(1/b-1)} \cdot G \tag{8}$$

$$\text{SMC}(Q) = (1/b) \cdot \mathbf{Q}^{(1/b-1)} \cdot G \tag{9}$$

Equations 7 and 9 show that SAVC and SMC are power functions of the same power, 1/b-1. Equation 8 is a rectangular hyperbola (SAFC = F/Q) added to SAVC. The output elasticity of labour, b, is less than 1 due to the law of diminishing returns so that SMC is 1/b > 1 time the size of SAVC at any output level (or, cross-multiplying we see that SAVC is b times the size of SMC). When b = 0.5, the linear SAVC is half as steep as SMC as is readily visible using the *CDCosts* sheet. The proportionality between SAVC and SMC remains true for other values of b. For example, when b = 0.75, SAVC and SMC are power functions of power 1/3 and SAVC is 3/4 the size of SMC at any output level. In Figure 8, SAVC(5; K_0=0.5) = \$3 and SMC(5; K_0=0.5) = \$4. SATC is also included in Figure 8 to show the essential nature of the relation of SATC, SMC, and SAVC. As always, SMC = SATC at minimum SATC (at approximately Q = 0.9). And, as Q increases, SATC approaches SAVC because as Q increases, AFC tends to zero. Formally, SATC is asymptotic to SAVC. Figure 8 is provided to show how this file can be used to examine these relations simply by modifying the sliders and click-boxes (in this instance, w was adjusted so that SMC and SAVC both cross the gridwork at an output level where the 3/4 nature of their relation is apparent given b = 0.75). It is instructive to use the w slider to note, for example, that as the wage rate increases the output level where SAVC = \$3 and SMC = \$4 decreases. Q = 4 (SAVC(4; K_0=0.5) = \$3 and SMC(4; K_0=0.5) = \$4) works when w = \$1.19, and Q = 3 (SAVC(3; K_0=0.5) = \$3 and SMC(3; K_0=0.5) = \$4) works when w = \$1.31.

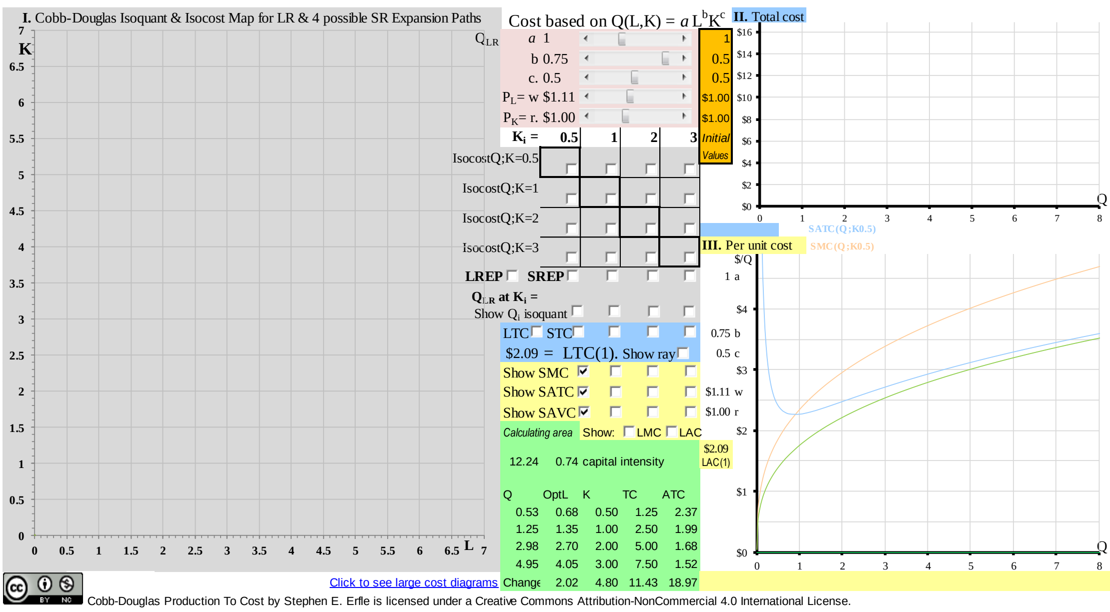

Figure 8: Screenshot of the *CDCosts* sheet to examine the relation of SMC and SAVC

3.2. CD LR costs

All factors of production are allowed to vary in the LR. As noted above, MRTS = w/r in the LR. The optimal capital for any given amount of labour is obtained by setting MRTS from Equation 2 equal to w/r and solving for K:

$$K = ((wc)/(rb))\cdot L \tag{10}$$

Substituting Equation 10 into Equation 1 provides output as a function of labour:

$$Q(L) = aL^b(((wc)/(rb))\cdot L)^c = a\cdot((wc)/(rb))^c\cdot L^{(b+c)} \tag{11}$$

Solving Equation 11 for L we obtain labour required to produce Q units of output in a cost-minimizing fashion as a function of the CD parameters and factor prices:

$$L = ((Q/a)\cdot((rb)/(wc))^c)^{(1/(b+c))} \tag{12}$$

Substituting Equation 12 into Equation 10 obtains the amount of capital required to produce Q units of output at minimum cost as a function of the CD parameters and factor prices:

$$K = ((Q/a)\cdot((wc)/(rb))^b)^{(1/(b+c))} \tag{13}$$

Equations 12 and 13 are derived (or conditional) factor demands. Multiplying Equation 12 times w and Equation 13 times r and adding them together, we obtain total cost as a function of output, CD production parameters, and factor prices:

$$LTC(Q) = Q^{(1/(b+c))}\cdot\mathbf{[}a^{(-1/(b+c))}\cdot w^{(b/(b+c))}\cdot r^{(c/(b+c))}\cdot((b/c)^{(c/(b+c)} + (c/b)^{(b/(b+c))})\mathbf{]} = \mathbf{Q^{(1/(b+c))}}\cdot H \tag{14}$$

As above, the second version of LTC focuses attention on the power function nature of LTC by simply noting that the bracketed term in the first version, H, is a function of the CD parameters together with factor prices, H = H(w, r, *a*, b, c). Note that H, unlike G in Equation 4, is not a function of K_0 but is a function of r (because this is a LR equation).

Per-unit costs are readily obtained from Equation 14. LAC is obtained by dividing LTC by Q and marginal cost is the derivative of LTC:

$$LAC(Q) = \mathbf{Q^{(1/(b+c)\ -\ 1)}}\cdot H \tag{15}$$

$$LMC(Q) = \mathbf{Q^{(1/(b+c)\ -\ 1)}}\cdot H/(b+c) \tag{16}$$

Equations 14-16 make clear that b+c determines the shape of each LR curve much as b determined the shape of each SR curve. When b+c = 1 production exhibits CRTS. LTC is a linear function of output and LAC = LMC is flat as noted in the geometric discussion in Section 2. If b+c > 1, we have IRTS in production. LTC is less than linear in Q, LAC and LMC are declining and LAC > LMC. And the reverse holds true when b+c < 1. In this instance, production exhibits DRTS, LTC increases more quickly than linearly (since 1/(b+c) > 1), LAC and LMC are increasing in Q (since 1/(b+c) – 1 > 0) and LAC < LMC.

The CD production function has some limitations. They cannot model varying returns to scale and factors of production have diminishing MP over the entire range of usage of the factors of production. This implies that STC curves are convex upward. LTC curves may be convex upward (if the production process exhibits DRTS), linear (if the production process exhibits CRTS) or upward sloping but convex downward (if the production process exhibits IRTS).

References

[1] Keat, P.G., Young, P.K.Y., and Erfle, S.E. (2013). *Managerial Economics: Economic Tools for Today's Decision Makers*, 7th Edition, 254, Boston, MA: Pearson.

[2] Varian, H. (2002). *Intermediate Microeconomics: A Modern Approach*, 6th Edition, 336, New York, NY: W. W. Norton.

[3] Pindyck, R.S., and Rubinfeld, D.L. (2013). *Microeconomics*, 8th Edition, 276–278, Upper Saddle River, NJ: Prentice Hall.

[4] Erfle, S.E. (2016). *Intermediate Microeconomics: An Interactive Approach*, Chapters 9, 10 and 11, Saint Paul, MN: Textbook Media Press.